Constitutive relations for steady, dense granular flows

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This work focuses on the mechanical response of dry granular materials under steady, simple shear conditions. In particular, the goal is to obtain a complete rheology able to describe the material behavior within the entire range of concentrations for which the flow can be considered dense. The total stress is assumed to be the linear sum of a frictional and a kinetic component. The frictional and the kinetic contributions are modeled in the context of the critical state theory and the kinetic theory of dense granular gases, respectively; in the latter, the correlated motion among the particles, which is likely to occur at high concentration, is also included. In accordance with recent findings on disordered granular packings, the frictional component of stresses is assumed to vanish when the concentration is below the random loose packing. According to this approach, four nondimensional quantities govern steady, simple shear flows: the concentration, the shear to normal stress ratio, the ratio of the time scales associated with the motion perpendicular and parallel to the flow, and the ratio between the particle stiffness and the normal stress. The present theory allows us to reproduce, in a notable way, both numerical simulations on simple shear flows of disks and physical experiments on incline flows of glass spheres taken from the literature.

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I. INTRODUCTION

Recently, the flow of dense granular materials has been the subject of many scientific works (Ref. [1] and references therein); this is mainly due to the large number of natural phenomena involving solid particles flowing at high concentration (e.g., debris flows and landslides) [2].

In contrast to the flow of dilute granular media, where the energy is essentially dissipated in binary collisions, the flow of dense granular materials is characterized by multiple, long-lasting, and frictional contacts among the particles. A satisfactory fundamental theory for the latter is still lacking, while the role of collisions has been successfully modeled by using the so-called kinetic theories [3–5].

The French research group GDR MiDi [6] has suggested that dense granular materials obey a local, phenomenological rheology that can be expressed in terms of two relations between three nondimensional quantities: concentration, shear to normal stress ratio, and the inertial parameter (ratio of the time scales associated with the motion perpendicular and parallel to the flow). Despite the notable results obtained in modeling many different configurations of dense granular flows [7–11], the GDR MiDi rheology does not apply when there is an additional time scale associated with the particle velocity fluctuations [12], whose intensity is provided by the granular temperature; in fact, the role of the latter cannot be disregarded in regions of thickness some diameters close to the boundaries (free surface, rigid, and/or erodible bottom) [13,14]. The so-called kinetic regime, characterized by widely spaced particles and rapid deformations, has been largely studied in the context of kinetic theories [5,15,16]. In those works, the particles are assumed to interact mainly through instantaneous, binary, and uncorrelated collisions. Jenkins [12,17] has recently extended the kinetic theories to account for the decrease in the energy dissipation due to the existence of correlated motion among the particles occurring at high concentration. Moreover, a heuristic extension of kinetic theories to deal with noninstantaneous interparticle collisions, due to the finite stiffness of the particles, has been suggested by Hwang and Hutter [18]. The possibility of sticking collisions has also been included in the theory in an approximate way [19]. Nevertheless, kinetic theories are not capable of capturing the roughly rate-independent behavior observed at concentrations near the random close packing [20].

On the other hand, a large number of constitutive relations have been proposed to account for the irreversible, nonholonomic, time-dependent mechanical behavior of granular media in the so-called quasistatic regime [21–25], i.e., when the deformations are extremely slow and the concentration approaches the random close packing. Those phenomenological constitutive models, mainly based on either the elastoplastic or the viscoplastic theory, do not incorporate the granular temperature as a state variable of the problem. Therefore, they are unable to deal with the phase transition of granular materials from a solid-like to a fluid-like state and vice versa.

The purpose of the present work is to propose general, physically sound constitutive equations for dense granular materials, including, as special cases, the aforementioned kinetic and quasistatic approaches. For the sake of simplicity, we focus on the case of steady, simple shear flows of identical, inelastic, frictional spheres, and we linearly add the kinetic (from the extended kinetic theory of Jenkins [12,17]) and the frictional contributions in the expression of the granular stresses.

The frictional component is modeled by using the "critical state" theory introduced in soil mechanics 50 years ago and still largely adopted [26,27]. According to this theory, the granular material approaches a certain attractor state, called the critical state, independent of the initial arrangement, characterized by the capability of a granular material of developing unlimited shear strains without any change in the concentration. There the shear stress results, proportional to the normal stress through the tangent of the critical friction angle, independent of the concentration. Also, the well-known critical state line, linking, in the semilogarithmic plane, the reciprocal of the concentration with the frictional normal stress, can be conveniently redefined on the basis of dimensional arguments in terms of a proportionality between the frictional normal stress and the particle stiffness through a function of the concentration. Given that a disordered granular packing, i.e., a dense granular material at zero granular temperature, exists only for a range of concentrations between the random loose and close packings, [28] a form for the concentration dependence of the frictional normal stress that makes the latter vanish at the random loose packing is defined. Very old experiments performed by Wroth [29] assess the validity of this assumption.

The idea of adding kinetic and frictional contributions in the granular stresses has been previously proposed by Johnson and Jackson [30,31]. However, Johnson and Jackson did not take into account the role of particle stiffness on the frictional component of the normal stress, so the constitutive relation for the latter was not physically based. Moreover, for the kinetic contribution to the stresses, they used a kinetic theory developed for dilute flows that does not take into account the breaking of the molecular chaos assumption [32] at high concentrations. Hence, they were unable to explain the variation of the ratio of shear to normal stress with concentration observed in numerical simulations on simple shear flows [20]. Similar considerations apply also to the theory developed by Savage [33], who assumed a plastic behavior and the presence of Gaussian fluctuations of the strain rate and stresses in the planar flow of a dense granular material and who obtained constitutive relations very similar to those of Johnson and Jackson [30,31].

The present theory is able to qualitatively and quantitatively reproduce both the numerical simulations on simple shear flows of disks obtained by da Cruz *et al.* [20] and the physical experiments on incline flows of glass spheres over rigid beds performed by Pouliquen [34].

This paper is organized as follows: Sec. II deals with the simplifying assumptions adopted in this work, the constitutive relations, and the analytical solution to the steady, simple shear flow of spheres at high concentration. In Sec. III, the theory is tested against the experimental and numerical results obtained by Pouliquen [34] and da Cruz *et al.* [20], and in Sec. IV, some concluding remarks and suggestions for future developments are drawn.

II. THEORY

A. Fundamental assumptions and constitutive relations

As already stated, we focus here on steady, simple shear flows of identical, inelastic, frictional, and cohesionless spheres. We make the fundamental assumption, as in Johnson and Jackson [30], that the kinetic and the frictional momentum exchange coexist in the flow and linearly add the two contributions in the expression of the granular stresses:

$$\begin{aligned} \sigma &= \sigma_k + \sigma_f, \\ \tau &= \tau_k + \tau_f, \end{aligned} \tag{1}$$

where σ and τ are the normal stress in the direction perpendicular to the flow and the shear stress, respectively (see Fig. 1 for a sketch of the flow configuration). Here and in what follows, all quantities are made dimensionless using the particle diameter

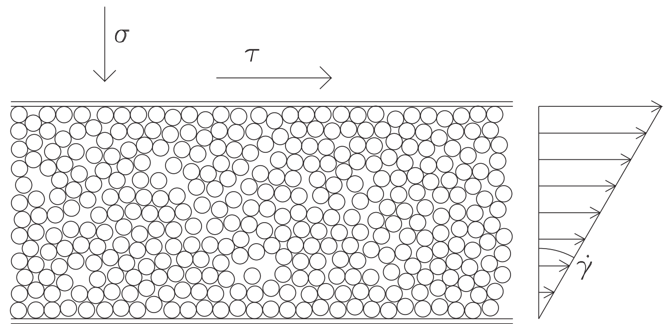

FIG. 1. Simple shear flow configuration.

and density and the gravitational acceleration. We use the subindices k and f to refer to quantities associated with the kinetic and the frictional contributions, respectively.

As already mentioned, we assume that at particle concentration ν below the random loose packing ν_{rlp} the frictional contribution to the stresses vanishes, in accordance with the range of existence of a disordered granular packing [28].

The constitutive relations proposed by Garzó and Dufty [35], as modified by Jenkins and Berzi [19], are employed to express the kinetic stresses,

$$\sigma_k = f_1 f_4 T, \tag{2}$$

$$\tau_k = f_2 f_4 T^{1/2} \dot{\gamma}, \tag{3}$$

and the rate of energy dissipation in collisions,

$$\Gamma = \frac{f_3}{L} f_4 T^{3/2}. \tag{4}$$

Here T is the granular temperature (one third of the mean square of the particle velocity fluctuations). The functions f_1, f_2, and f_3 in the dense limit, i.e., for concentrations higher than, say, 40%, are derived from those reported in [19] and summarized in Table I. As in [19,36], e is an effective coefficient of restitution that depends on the coefficient of normal restitution ϵ and the coefficient of tangential restitution in a sticking collision β [37]. G is the product of ν and the radial distribution function; for the latter, the expression

TABLE I. Auxiliary functions for the kinetic constitutive relations.

$f_1 =$	$2(1+e)\nu G$
$f_2 =$	$\dfrac{8J}{5\pi^{1/2}}\nu G$
$f_3 =$	$\dfrac{12}{\pi^{1/2}}\nu G(1-e^2)$
$f_4 =$	$\left(1+\dfrac{2}{s}\dfrac{T^{1/2}}{E^{1/2}}\right)^{-1}$
$\dfrac{1-e^2}{4} \equiv$	$\dfrac{1-\epsilon^2}{4}+\dfrac{1+\beta}{7}-\left(\dfrac{1+\beta}{7}\right)^2\left[1+\dfrac{5(1+\beta)}{14-5(1+\beta)}\right]$
$G =$	$\dfrac{5.69\nu(\nu_{\mathrm{rcp}}-0.49)}{\nu_{\mathrm{rcp}}-\nu}$
$J =$	$\dfrac{1+e}{2}+\dfrac{\pi}{4}\dfrac{(3e-1)(1+e)^2}{[24-(1-e)(11-e)]}$
$L =$	$\max\left[1,\left(\dfrac{1}{2}c\dfrac{G^{1/3}}{T^{1/2}}\dot{\gamma}\right)\right]$

suggested by Torquato [38], which diverges as the concentration approaches the random close packing, $\nu_{\rm rcp}$, is adopted. In Eq. (4), L is a correlation length, accounting for the decrease in the collisional energy dissipation due to the presence of correlated motion of particles that is likely to occur when the flow is dense [12,17]. In its expression, reported in Table I, c is a constant of order unity.

The function f_4 in Eqs. (2)–(4), not present in the constitutive relations of Jenkins and Berzi [19], takes into account the influence of the particle stiffness on the contact duration during collisions; its form, reported in Table I, has been suggested by Hwang and Hutter [18]. There E is the Young's modulus of the particles, and s is the mean separation distance between the particles (at high concentration, it is of the order of one tenth of a diameter).

As already mentioned, the frictional component of the shear stress is taken to be proportional to the frictional component of the normal stress through the tangent of the critical friction angle ϕ_c, [26,27]

$$\tau_f = \sigma_f \tan\phi_c. \tag{5}$$

Dimensional analysis suggests that the constitutive relation for the frictional normal stress in the absence of flow must be written as

$$\sigma_f = f_5 K, \tag{6}$$

where K is the particle stiffness, equal to $\pi E/8$ in the case of linear contacts [39], and f_5 is solely a function of the concentration. σ_f is expected to increase with concentration and diverge at the random close packing. Since σ_f is also assumed to vanish at the random loose packing, we take

$$f_5 = \max\left(a\frac{\nu - \nu_{\rm rlp}}{\nu_{\rm rcp} - \nu}, 0\right), \tag{7}$$

where a is a material coefficient inferred from experiments.

Experimental investigations on the critical state of identical spheres are, however, rare. To our knowledge, only Wroth [29] performed experiments on the critical state of 1 mm stainless spheres [40]. The experiments confirm that the ratio of τ_f to σ_f, i.e., $\tan\phi_c$, is independent of the concentration and that f_5 is a unique function of ν (Fig. 2). There the steel Young's modulus is known, giving $K = 1.1 \times 10^9$. The solid line in Fig. 2 corresponds to the theoretical expression of Eq. (7), with $\nu_{\rm rcp} = 0.619$, $\nu_{\rm rlp} = 0.598$, and $a = 1.8 \times 10^{-6}$ (obtained from linear regression).

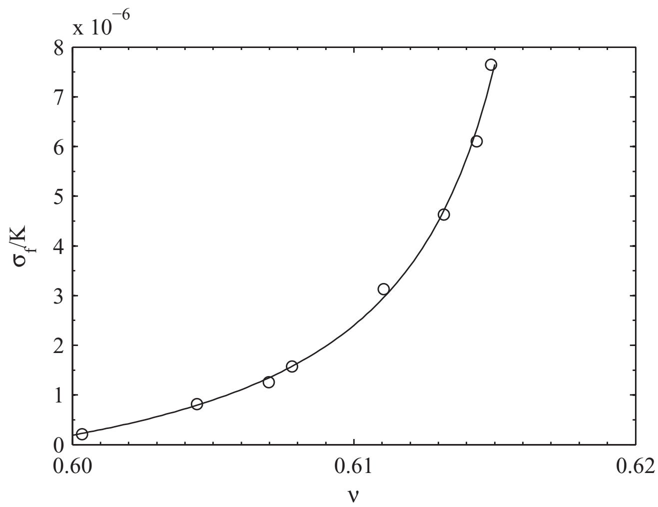

FIG. 2. Experimental (circles, from Ref. [29]) against theoretical [solid line, from Eq. (7)] coefficient $f_5 = \sigma_f/K$ as function of concentration for steel spheres.

B. Analytical solution of steady, simple shear flow

Under the usual assumptions of constant shear and normal stresses, the steady, simple shear flow is completely governed by the balance of fluctuating energy,

$$\tau_k \dot{\gamma} = \Gamma. \tag{8}$$

As the granular temperature is constant in the steady, simple shear flow, the divergence of the flux of fluctuating energy is here neglected. Moreover, as in Johnson and Jackson [30], the two dissipation mechanisms (i.e., collisions and friction) are assumed to be uncoupled, and the work of the frictional component of the shear stress is postulated not to produce fluctuating energy. Hence, Eq. (8) implies that the energy produced through the work of the kinetic shear stress is entirely dissipated in collisions.

Under this fundamental hypothesis, by substituting Eqs. (3) and (4) in Eq. (8) and using the constitutive expression for L of Table I, there is an algebraic relation between the shear rate and the granular temperature,

$$T = f_6 \dot{\gamma}^2, \tag{9}$$

with $f_6 = f_2 L/f_3$. Equation (9) shows that the time scale associated with the particle fluctuations is not a free variable in steady, simple shear flows but can be algebraically obtained from the time scale associated with the shear rate, as already shown by Jenkins [12] and Jenkins and Berzi [19]. In this situation, the dimensional analysis performed by the French research group GDR MiDi [6] holds, and the flow can be completely described by three dimensionless quantities if the particles are rigid: the inertial number $I \equiv \dot{\gamma}(\nu/\sigma)^{1/2}$, the concentration ν, and the stress ratio $\mu \equiv \tau/\sigma$. In particular, as already mentioned, the inertial number represents the ratio between the microscopic time scale $(\nu/\sigma)^{1/2}$, associated with the transversal motion of a particle submitted to a normal stress σ, and the macroscopic time scale $1/\dot{\gamma}$, associated with the motion parallel to the flow. da Cruz *et al.* [20] suggest that for small values of I, i.e., small shear rate and/or large pressure, the particle inertia is not relevant and the flow is in the quasistatic regime, whereas large values of I correspond to the kinetic regime.

Using Eqs. (1)–(3), (5), (6), and (9), the constitutive relations of Table I, and the definition of the inertial number, we obtain

$$\mu = \tan\phi_c + \left(f_2 f_6^{1/2} - \tan\phi_c f_1 f_6\right) f_4 \frac{I^2}{\nu} \tag{10}$$

and

$$I = \left[\frac{\nu}{f_1 f_6 f_4}\left(1 - \frac{K f_5}{\sigma}\right)\right]^{1/2}. \tag{11}$$

By setting $I = 0$, i.e., $\dot{\gamma} = 0$, in Eq. (11), the maximum concentration that can be achieved under steady conditions reads

$$\nu_{\max} = \frac{\nu_{\mathrm{rcp}}}{1 + aK/\sigma} + \frac{\nu_{\mathrm{rlp}}}{1 + \sigma/(aK)}. \qquad (12)$$

For small values of K/σ, $\nu_{\max}$ approaches the random close packing; on the other hand, $\nu_{\max}$ tends to the random loose packing when K/σ is large.

In this work, we limit the analysis to values of K/σ for which the coefficient f_4 is approximately 1; i.e., the particle stiffness does not affect the kinetic components of the stresses. This is mainly due to fact that an analytical expression for the mean separation distance between the particles is still lacking. The coefficient f_4 of Table I can be rewritten as

$$f_4 = \left[1 + 2\left(\frac{\pi}{8}\right)^{1/2} \frac{\nu^{-1/2}}{s} f_6^{1/2} I \left(\frac{K}{\sigma}\right)^{-1/2}\right]^{-1}. \qquad (13)$$

For dense flows, s and f_6 are of the order of 10^{-1}, while ν and I are of the order of unity. The second member on the right hand side of Eq. (13) can therefore be neglected if K/σ is greater than 10^5.

III. RESULTS AND DISCUSSION

In this section, the theory is validated against the numerical simulations on steady, simple shear flows of slightly polydispersed, cohesionless disks [20] and the experiments on incline flows of glass spheres over rigid beds [34].

In the following, we use $a = 1.8 \times 10^{-6}$, $\nu_{\mathrm{rlp}} = 0.598$, and $\nu_{\mathrm{rcp}} = 0.619$, as for steel spheres; $e = 0.60$ and $c = 0.50$, as suggested by Jenkins [12] and Jenkins and Berzi [19] for dense flows of glass spheres; and $\tan\phi_c = 0.38$, the angle of repose for glass spheres reported by Pouliquen [34].

Figure 3 shows the theoretical relations of Eqs. (10) and (11) between the stress ratio μ, the concentration ν, and the inertial number I for three different ratios K/σ. All the distinctive features observed by da Cruz *et al.* [20] on numerical simulations on disks are present in Fig. 3: (i) At the lowest values of I, the kinetic components of the stresses are negligible, so the stress ratio is approximately constant and equal to $\tan\phi_c$, a substantially rate-independent regime [Fig. 3(a)]. (ii) In that regime, the concentration [Fig. 3(b)] shows the tendency to saturate toward $\nu_{\max}$, lying between the random loose and close packings. (iii) At the largest values of I, the frictional components of the stresses vanish, and the stress ratio saturates to a constant value, as predicted by classical kinetic theories in the dense limit [41].

Figure 3 shows that K/σ does not substantially affect the curves since, for the granular material here considered, the values of ν_{rlp} and ν_{rcp} are very close. A deeper investigation of the influence of K/σ for values smaller than 10^5, i.e., for softer particles and/or for larger normal stresses (of interest in earth science), is deferred to future presentations.

Also shown in Fig. 3(a) are the experimental results on the steady and fully developed flows of glass spheres on inclined planes performed by Pouliquen [34], for which K/σ is of the order of 10^8. In that flow configuration, the stress ratio is constant along the flow cross section and equal to the tangent of the angle of inclination of the plane [19]. If the flow is thick enough (say, depths greater than ten diameters), the influence of the boundaries can be neglected, and both the inertial number and the concentration are also constant along the cross section of the flow [13,19]. The incline flow configuration works therefore as a rheometer [6] and provides values of μ and I that can be compared with those derived from the present theory.

Pouliquen [34] measured the particle depth-averaged velocity V and the depth h for different angles of inclination of the plane θ (ranging from 22° to 28°). As shown by GDR MiDi [6], the experimental values of the inertial number and the stress ratio correspond to $I = 5V/[2(\cos\theta)^{1/2}h^{3/2}]$ and $\mu = \tan\theta$, respectively.

The experimental values reported on Fig. 3(a) have been obtained by averaging all the data reported by Pouliquen [34] with depths greater than ten diameters. The agreement between the theoretical and the experimental results is remarkable, especially because there is no tuning of the model parameters. It is also worth emphasizing that the constitutive relation for the frictional component of the normal stress holds, in

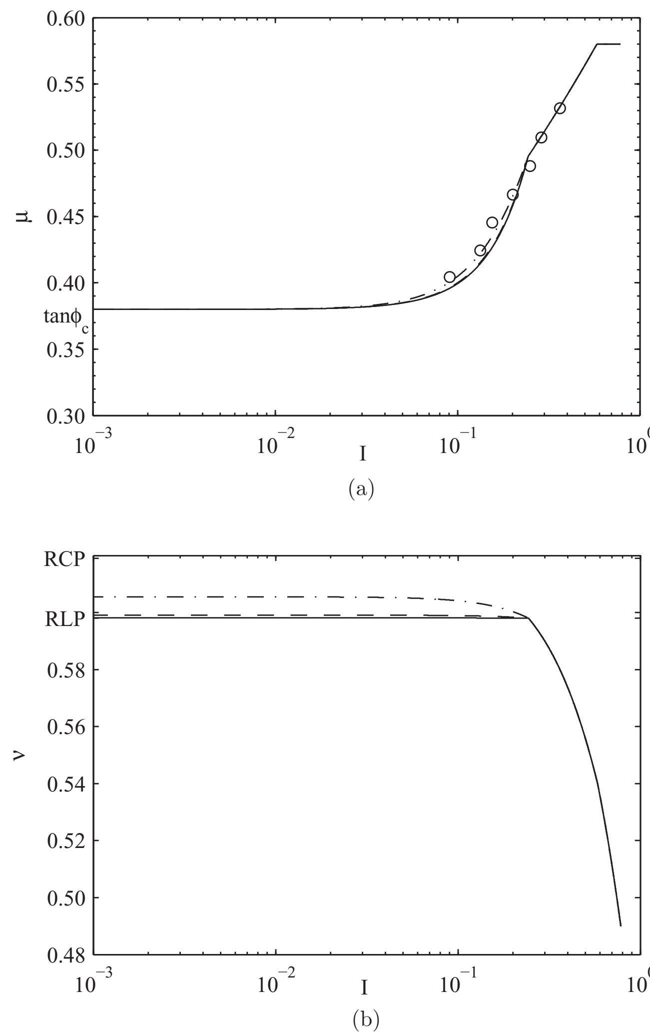

FIG. 3. Theoretical (a) stress ratio and (b) concentration as functions of the inertial number for $K/\sigma = 10^6$ (dot-dashed line), $K/\sigma = 10^7$ (dashed line), and $K/\sigma = 10^8$ (solid line).

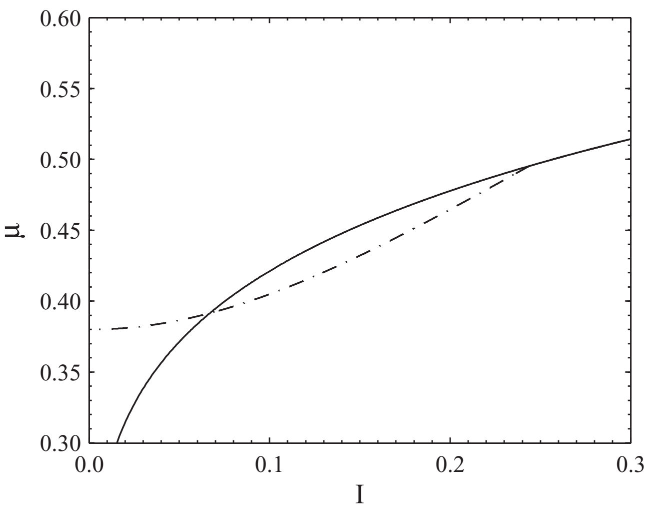

FIG. 4. Theoretical stress ratio as a function of the inertial number for frictional (solid line) and frictionless (dash-dotted line) particles.

principle, for steel spheres. As already stated, the theories of Johnson and Jackson [30,31] and Savage [33] would predict a constant stress ratio, independent of the inertial number, in contrast to the experiments. da Cruz *et al.* [20] also investigated the influence of the interparticle friction on the curves of Fig. 3. The interparticle friction affects the critical friction angle [20] and the values of ν_{rlp} and ν_{rcp} [28,39]. In particular, for frictionless particles, $\tan\phi_c = 0$ and $\nu_{\mathrm{rlp}} = \nu_{\mathrm{rcp}} = 0.634$ [28,42]. Figure 4 shows that, unlike the frictional case, the present theory predicts a strictly sublinear dependence of the stress ratio on the inertial number for frictionless particles, a distinctive feature observed in the numerical simulations [20].

IV. CONCLUSIONS

In this paper, a constitutive model for dense granular flows of inelastic, frictional, identical spheres, obtained by linearly adding the frictional and the kinetic stresses, is proposed. We have assumed that the frictional component of the stress vanishes when the particle concentration is less than the random loose packing, which represents the lower limit for the existence of a disordered granular packing [28]. Both the role of particle stiffness and the correlated motion among the particles have been accounted for. We have used the constitutive relations to solve for the steady, simple shear flow of spheres, and we have shown that the present theory is capable of reproducing, qualitatively and quantitatively, the numerical simulations on disks [20] and the experiments on incline flows of glass spheres [34]. We have restricted the analysis to flows for which the ratio of the particle stiffness to the normal stress is large enough to not affect the kinetic components of the stresses. In fact, many geophysical flows are characterized by lower values of that ratio, so further investigations are required. An extension of the present approach to deal with unsteady problems will be the subject of future presentations.

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